③ Formal Theory of Angular-Momentum Addition

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \qquad \| \ [J_{1i}, J_{2j}] = 0$$

↳ It's also a generator of a rotation.

∘ Infinitesimal rotation

$$1 - \frac{i}{\hbar}(\vec{J}_1 \otimes I + I \otimes \vec{J}_2)\cdot \hat{n}\,\delta\phi = \left(1 - \frac{i}{\hbar}\vec{J}_1\cdot\hat{n}\,\delta\phi\right) \otimes \left(1 - \frac{i}{\hbar}\vec{J}_2\cdot\hat{n}\,\delta\phi\right)$$

$$\Leftrightarrow \quad \mathcal{D}_1(R) \otimes \mathcal{D}_2(R) = \mathcal{D}_{1+2}(R)$$

Since $\vec{J}_1$, $\vec{J}_2$ and $\vec{J}$ belong to the same group,

$$[J_i, J_j] = i\hbar\varepsilon_{ijk} J_k \quad \rightarrow \quad \boxed{\begin{aligned} J^2|j,m\rangle &= j(j+1)\hbar^2|j,m\rangle \\ J_z|j,m\rangle &= m\hbar|j,m\rangle \end{aligned}}$$

Verification

$$\begin{aligned}
J_i J_j - J_j J_i &= (J_{1i}\otimes I + I\otimes J_{2i})(J_{1j}\otimes I + I\otimes J_{2j}) \\
&\quad - (J_{1j}\otimes I + I\otimes J_{2j})(J_{1i}\otimes I + I\otimes J_{2i}) \\
&= (J_{1i}J_{1j})\otimes I + I\otimes(J_{2i}J_{2j}) + \cancel{J_{1j}\otimes J_{2i}} + \cancel{J_{1i}\otimes J_{2j}} \\
&\quad - (J_{1j}J_{1i})\otimes I - I\otimes(J_{2j}J_{2i}) - \cancel{J_{1j}\otimes J_{2i}} - \cancel{J_{1i}\otimes J_{2j}} \\
&= [J_{1i}, J_{1j}]\otimes I + I\otimes[J_{2i}, J_{2j}] \\
&= i\hbar\,\varepsilon_{ijk}(J_{1k}\otimes I + I\otimes J_{2k}) \\
&= i\hbar\,\varepsilon_{ijk} J_k .
\end{aligned}$$

The Goal: To find a systematic way

to connect $|j, m\rangle$ and $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$.

- notations of eigenkets.

a. $|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |j_1 j_2 ; m_1 m_2\rangle$

In some other books, $|j_1 m_1 ; j_2 m_2\rangle$ is preferred.

b. $|j, m\rangle$: Is (j, m) good enough?

Do they make a complete set of mutually commuting observables?

(J^2, J_z)

No! $[\vec{J}^2, \vec{J}_1^2] = 0$, $[\vec{J}^2, \vec{J}_2^2] = 0$

but, $[\vec{J}^2, J_{1z}] \neq 0$, $[\vec{J}^2, J_{2z}] \neq 0$.

Verification

NOTE: $[J_{1i}, J_{2j}] = 0$

$$J^2 = J_1^2 + J_2^2 + 2\vec{J}_1 \cdot \vec{J}_2$$

$$= J_1^2 + J_2^2 + 2J_{1z}J_{2z} + 2 \cdot \frac{1}{2}\frac{1}{2}(J_{1+} + J_{1-})(J_{2+} + J_{2-})$$

$$+ 2 \cdot \frac{1}{2i} \cdot \frac{1}{2i}(J_{1+} - J_{1-})(J_{2+} - J_{2-})$$

$$= J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

since $[J_1^2, J_{1z}] = 0$, $[J_1^2, J_{1\pm}] = 0$.

$\rightarrow [J^2, J_1^2] = 0$ and similarly, $[J^2, J_2^2] = 0$.

$\Rightarrow$ The eigenket can be written as

$|j_1 j_2 ; j\, m\rangle$ for $\vec{J} = \vec{J}_1 + \vec{J}_2$,

since the complete set of commuting observables

is $\{J^2, J_z, J_1^2, J_2^2\}$.

• Clebsch – Gordan Coefficients

Consider a change of base kets : $|j_1 j_2 ; m_1 m_2\rangle \longrightarrow |j_1 j_2 ; j m\rangle$

Using the completeness $\sum_{m_1, m_2} |j_1 j_2 ; m_1 m_2\rangle\langle j_1 j_2 ; m_1 m_2| = 1$,

$$\rightarrow |j_1 j_2 ; jm\rangle = \sum_{m_1, m_2} |j_1 j_2 ; m_1 m_2\rangle \langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; jm\rangle$$

$\begin{pmatrix} \text{(orthogonal)} \\ \text{unitary} \\ \text{matrix.} \end{pmatrix} \leftarrow$ Clebsch – Gordan Coeff.

$\equiv C^{j_1 j_2}_{m_1 m_2 ; jm}$ in some books.

* The properties of C–G coeffs.

① $C^{j_1 j_2}_{m_1 m_2 ; jm} = 0$ unless $m = m_1 + m_2$ ☆

proof. use $J_z = J_{1z} + J_{2z}$.

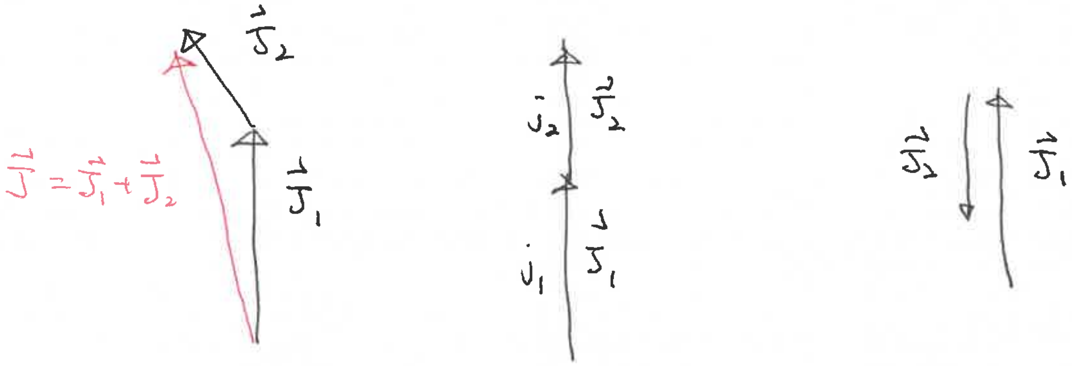
$\hookrightarrow \langle j_1 j_2 ; m_1 m_2 | (J_z - J_{1z} - J_{2z}) | j_1 j_2 ; jm\rangle = 0.$

$$\Rightarrow (m - m_1 - m_2)\, C^{j_1 j_2}_{m_1 m_2 ; jm} = 0.$$

② $C^{j_1 j_2}_{m_1 m_2 ; jm} = 0$ unless $|j_1 - j_2| \le j \le j_1 + j_2$ ☆

- a hand-waving way to see it : a vector sum.

• Maximum length • minimum length



$\Rightarrow j_{max} = j_1 + j_2$ $\Rightarrow j_{min} = |j_1 - j_2|$

proof.

Ref. Le Bellac 10.6

- degeneracy of the eigenvalue m of J_z :

$$\boxed{n(m) = \sum_{j \geq |m|} N(j)}$$

ex.) $1 \otimes \frac{1}{2}$: $n(\frac{1}{2}) = N(\frac{1}{2}) + N(\frac{3}{2}) = 2$

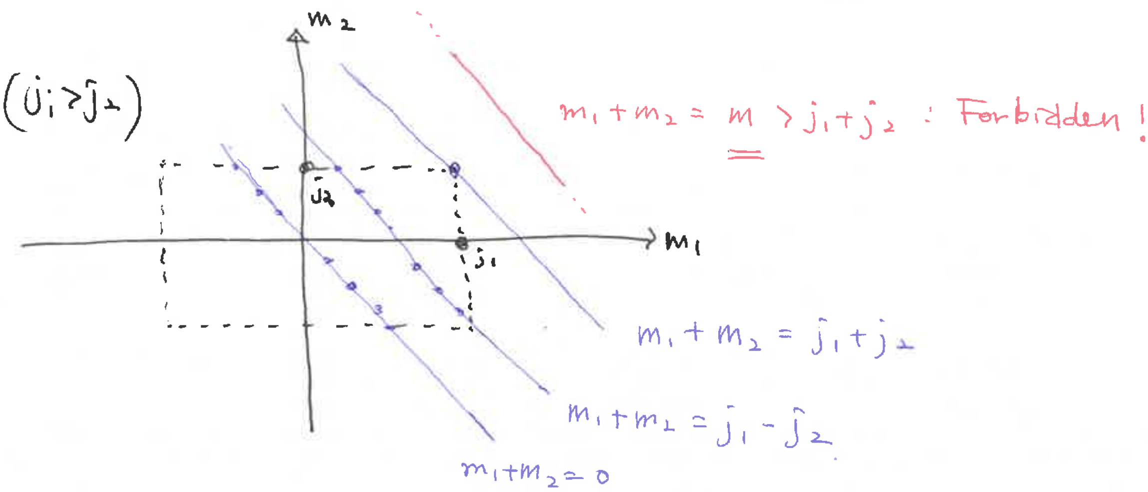
. $j = \frac{3}{2}$: $m = +\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$

. $j = \frac{1}{2}$: $m = \frac{1}{2}, -\frac{1}{2}$

$$\Rightarrow \boxed{N(x) = n(x) - n(x+1)}$$

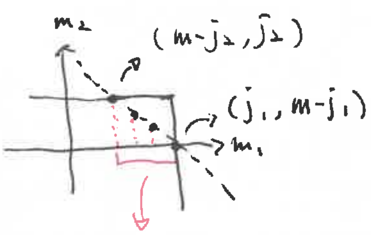
$$= [\cdots + \cancel{N(x+1)} + N(x)] - [\cdots + \cancel{N(x+1)}]$$

- counting degeneracy in (m_1, m_2) - space : $m = m_1 + m_2$

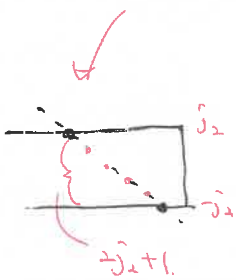


$n(m)$ = number of grid points in ⬚ .

$$= \begin{cases} 0 & \text{if } |m| > j_1 + j_2 \\ j_1 + j_2 + 1 - |m| & \text{if } |j_1 - j_2| \leq m \leq j_1 + j_2 \\ 2j_2 + 1 & \text{if } 0 \leq |m| \leq j_1 - j_2 \end{cases}$$



$j_1 - (m - j_2) + 1 = j_1 + j_2 + 1 - m$



$\therefore N(j) = 1$

for $|j_1 - j_2| \leq j \leq j_1 + j_2$

\#

• The arbitrariness of the overall phase : just set to be REAL

$$C^{j_1 j_2\,*}_{m_1 m_2 ; jm} = C^{j_1 j_2}_{m_1 m_2 ; jm} \to \text{orthogonal matrix.}$$

or $\langle j_1 j_2 ; jm | j_1 j_2 ; m_1 m_2 \rangle = \langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; jm \rangle \equiv C^{j_1 j_2}_{m_1 m_2 ; jm}$

→ orthogonality condition :

$$\sum_j \sum_m C^{j_1 j_2}_{m_1 m_2 ; jm} C^{j_1 j_2}_{m_1' m_2' ; jm} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

and

$$\sum_{m_1} \sum_{m_2} C^{j_1 j_2}_{m_1 m_2 ; jm} C^{j_1 j_2}_{m_1 m_2 ; j' m'} = \delta_{jj'} \delta_{mm'}$$

special case : $j' = j$, $m' = m = m_1 + m_2$

$$\to \sum_{m_1} \sum_{m_2} \left[C^{j_1 j_2}_{m_1 m_2 ; jm} \right]^2 = \sum_{m_1, m_2} \left| \langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; jm \rangle \right|^2 = 1 \,,$$

: normalization condition of $|j_1 j_2 ; jm\rangle$.

• Another notation of the CG coeff.

$$C^{j_1 j_2}_{m_1 m_2 ; jm} = (-1)^{j_1 - j_2 + m} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$$

Wigner's 3-j symbol.

$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \Rightarrow$ [invariant under cyclic perm. ; $(-1)^{j_1+j_2+j_3}$ · (non-cyclic perm.)]

(see Commins 7.8)

There are symmetry relations for the Wigner's 3-j symbol.

* general formula

by Wigner (1959)
by Schwinger (1942)
by Racah

$$C^{j_1 j_2}_{m_1 m_2 ; jm} = \delta_{m_1+m_2, m} \left[\frac{(2j+1)(j_1+j_2-j)!\,(j_1-j_2+j)!\,(-j_1+j_2+j)!}{(j_1+j_2+j_3+1)!} \right]^{\frac{1}{2}}$$

$$\cdot \sum_n (-1)^n \frac{\left[(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\,(j+m)!\,(j-m)! \right]^{\frac{1}{2}}}{n!\,(j_1+j_2-j-n)!\,(j_1-m_1-n)!\,(j_2+m_2-n)!\,(j-j_2+m_1+n)!\,(j-j_1-m_2+n)!}$$

④ Recursion Relations for the C-G coeffs.

$$J_\pm |j_1 j_2 ; j m\rangle = (J_{1\pm} + J_{2\pm}) \sum_{m_1' m_2'} C^{j_1 j_2}_{m_1' m_2' ; j m} |j_1 j_2 ; m_1' m_2'\rangle$$

$\downarrow$

$$\sqrt{(j \mp m)(j \pm m + 1)}\, |j_1 j_2 ; j, m \pm 1\rangle$$

$$= \sum_{m_1' m_2'} \left(\sqrt{(j_1 \mp m_1')(j_1 \pm m_1' + 1)}\, |j_1 j_2 ; m_1' \pm 1, m_2'\rangle + \sqrt{(j_2 \mp m_2')(j_2 \pm m_2' + 1)}\, |j_1 j_2 ; m_1', m_2' \pm 1\rangle \right) \cdot C^{j_1 j_2}_{m_1' m_2' ; j m}$$

multiplying $\langle j_1 j_2 ; m_1, m_2 | \cdot$

orthogonality : $m_1 = m_1' \pm 1$, $m_2 = m_2'$

$$\sqrt{(j \mp m)(j \pm m + 1)}\, C^{j_1 j_2}_{m_1 m_2 ; j, m \pm 1}$$

$$= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)}\, C^{j_1 j_2}_{m_1 \mp 1, m_2 ; j m}$$

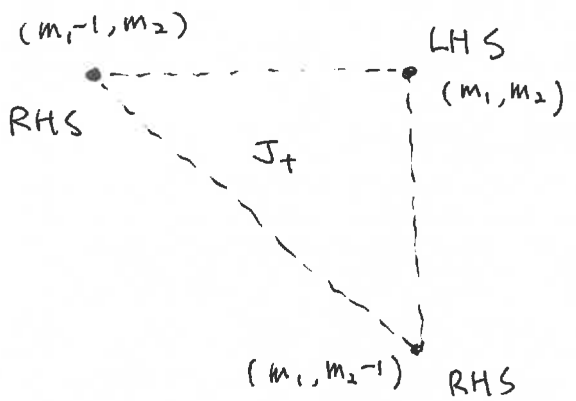
$$+ \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)}\, C^{j_1 j_2}_{m_1, m_2 \mp 1 ; j m}$$

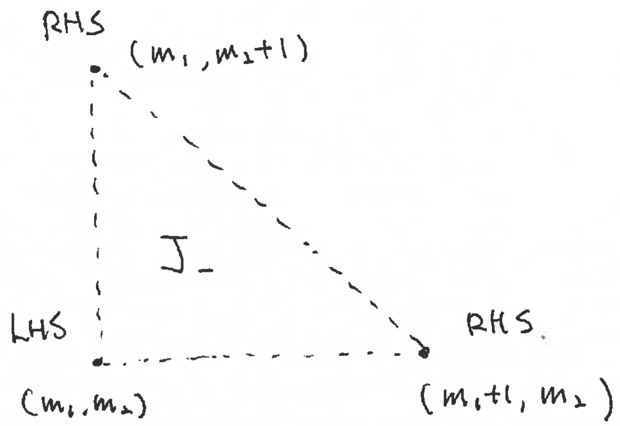
$m_1 = m_1'$
$m_2 = m_2' \pm 1$

☆☆

$\longrightarrow$ Recursion relation for the CG coeffs. .... eq. (*)

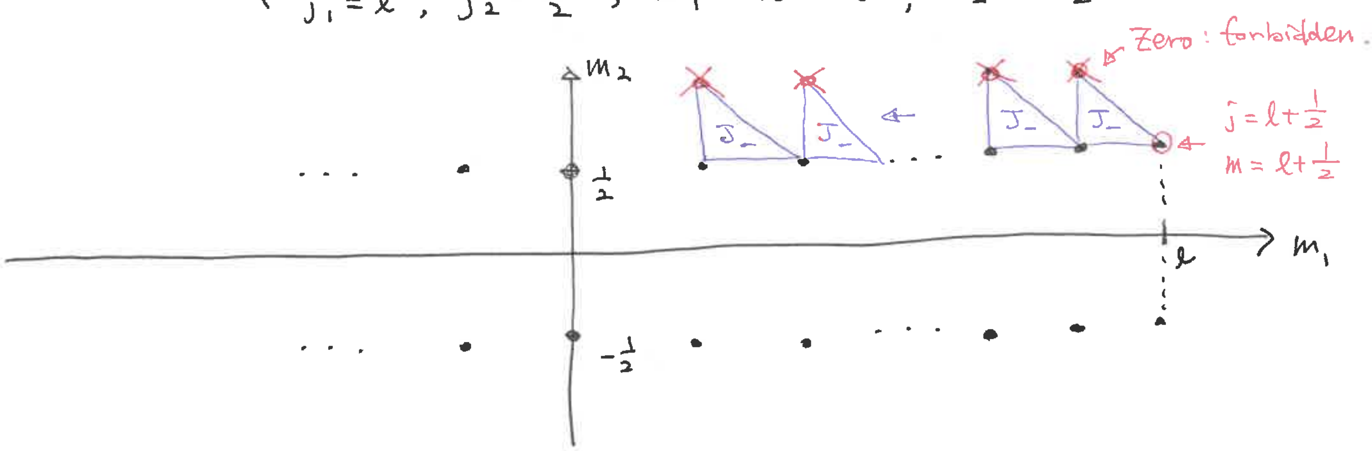
What it says :





* Exercise : $\vec{J} = \vec{L} + \vec{S}$

$(j_1 = \ell \,,\ j_2 = \frac{1}{2} \,,\ m_1 = -\ell \cdots \ell \,,\ m_2 = \pm\frac{1}{2})$



• Let's start from the upper-right corner.

$(j = \ell + \frac{1}{2} ,\ m = \ell + \frac{1}{2} ,\ m_1 = \ell ,\ m_2 = \frac{1}{2})$

$$\text{RHS} = C^{\ell,\frac{1}{2}}_{\ell,\frac{1}{2};\ \ell+\frac{1}{2},\ \underbrace{\ell+\frac{1}{2}}_{m}} \sqrt{(\ell + (\ell-1) + 1)(\ell - (\ell-1))}$$

$$\text{LHS} = C^{\ell,\frac{1}{2}}_{\underbrace{\ell-1}_{m_1},\frac{1}{2};\ \ell+\frac{1}{2},\ \ell-\frac{1}{2}} \sqrt{(\ell+\tfrac{1}{2} + (\ell+\tfrac{1}{2}))(\ell+\tfrac{1}{2} - (\ell+\tfrac{1}{2}) + 1)}$$

$$\Rightarrow \quad C^{\ell,\frac{1}{2}}_{\ell-1,\frac{1}{2};\ \ell+\frac{1}{2},\ \ell-\frac{1}{2}} = \frac{\sqrt{2\ell}}{\sqrt{2\ell+1}} \, C^{\ell,\frac{1}{2}}_{\ell,\frac{1}{2};\ \ell+\frac{1}{2},\ \ell+\frac{1}{2}} \quad //$$

• From $m_1 = m_\ell + 1$ to $m_1 = m_\ell$:

$\bullet \xleftarrow{J_-} \bullet$

$\begin{pmatrix} m_1 = m_\ell \\ m_2 = \frac{1}{2} \end{pmatrix} \quad \begin{pmatrix} m_1 = m_\ell + 1 \\ m_2 = \frac{1}{2} \end{pmatrix}$

$m_J = m_\ell + \frac{3}{2}$

$$\text{RHS} = C^{\ell,\frac{1}{2}}_{m_\ell+1,\frac{1}{2};\ \ell+\frac{1}{2},\ m_J} \sqrt{(\ell + m_\ell + 1)\cancel{(\ell - m_\ell)}}$$

$$\text{LHS} = C^{\ell,\frac{1}{2}}_{m_\ell,\frac{1}{2};\ \ell+\frac{1}{2},\ m_J-1} \sqrt{(\ell+\tfrac{1}{2} + m_J)\cancel{(\ell+\tfrac{1}{2} - m_J + 1)}}$$

If we set $m = m_J - 1$, $m_\ell = m - \frac{1}{2}$

$$\Rightarrow \quad C^{\ell,\frac{1}{2}}_{m-\frac{1}{2},\frac{1}{2};\ \ell+\frac{1}{2},\ m} = \frac{\sqrt{\ell + m + \frac{1}{2}}}{\sqrt{\ell + m + \frac{3}{2}}} \, C^{\ell,\frac{1}{2}}_{m+\frac{1}{2},\frac{1}{2};\ \ell+\frac{1}{2},\ m+1}$$

$$\Rightarrow C^{l,\frac{1}{2}}_{m-\frac{1}{2},\frac{1}{2};l+\frac{1}{2},m} = \frac{\sqrt{l+m+\frac{1}{2}}}{\sqrt{l+m+\frac{3}{2}}} C^{l,\frac{1}{2}}_{m+\frac{1}{2},\frac{1}{2};l+\frac{1}{2},m+1}$$

$$= \frac{\sqrt{l+m+\frac{1}{2}}}{\cancel{\sqrt{l+m+3/2}}} \cdot \frac{\cancel{\sqrt{l+m+3/2}}}{\sqrt{l+m+5/2}} C^{l,\frac{1}{2}}_{m+\frac{3}{2},\frac{1}{2};l+\frac{1}{2},m+2}$$

$$= \sqrt{l+m+1/2} \cdots \frac{\cancel{\sqrt{2l}}}{\sqrt{2l+1}} C^{l,\frac{1}{2}}_{l,\frac{1}{2};l+\frac{1}{2},l+\frac{1}{2}}$$

$$\Rightarrow C^{l,\frac{1}{2}}_{m-\frac{1}{2},\frac{1}{2};l+\frac{1}{2},m} = \frac{\sqrt{l+m+1/2}}{\sqrt{2l+1}} C^{l,\frac{1}{2}}_{l,\frac{1}{2};l+\frac{1}{2},l+\frac{1}{2}}$$

still, this is unknown.

Choose "1" by convention.

Thus, for $|j_1 j_2; j m\rangle = \sum_{m_1 m_2} C^{j_1 j_2}_{m_1 m_2; jm} |j_1 j_2; m_1 m_2\rangle$,

$$|l,\tfrac{1}{2};l+\tfrac{1}{2},m\rangle = \sqrt{\frac{l+m+1/2}{2l+1}}\, |l,\tfrac{1}{2};m-\tfrac{1}{2},\tfrac{1}{2}\rangle$$

We now have one coefficient!

$$+ C^{l,\frac{1}{2}}_{m+\frac{1}{2},-\frac{1}{2};l+\frac{1}{2},m} \;??? \; |l,\tfrac{1}{2};m+\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

We still do not know it.

also,

$$|l,\tfrac{1}{2};l-\tfrac{1}{2},m\rangle = C^{l,\frac{1}{2}}_{m-\frac{1}{2},\frac{1}{2};l-\frac{1}{2},m} \;???\; |l,\tfrac{1}{2};m-\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$+ C^{l,\frac{1}{2}}_{m+\frac{1}{2},-\frac{1}{2};l-\frac{1}{2},m} \;???\; |l,\tfrac{1}{2};m+\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

The three unknowns can be determined by the orthogonality of C-G coefficients.

$$\begin{pmatrix} |l,\tfrac12; l+\tfrac12, m\rangle \\ |l,\tfrac12; l-\tfrac12, m\rangle \end{pmatrix} = \underbrace{\begin{pmatrix} C^{l,\frac12}_{m-\frac12,\frac12;\,l+\frac12,m} & C^{l,\frac12}_{m+\frac12,-\frac12;\,l+\frac12,m} \\ C^{l,\frac12}_{m-\frac12,\frac12;\,l-\frac12,m} & C^{l,\frac12}_{m+\frac12,-\frac12;\,l-\frac12,m} \end{pmatrix}}_{2\times 2 \text{ orthogonal matrix}} \begin{pmatrix} |l,\tfrac12; m-\tfrac12,\tfrac12\rangle \\ |l,\tfrac12; m+\tfrac12,-\tfrac12\rangle \end{pmatrix}$$

$$\equiv \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},$$

$\rightarrow \sin^2\alpha = 1 - \dfrac{l+m+\frac12}{2l+1}$

$= \dfrac{l-m+\frac12}{2l+1}$

$$\begin{pmatrix} \sqrt{\dfrac{l+m+\frac12}{2l+1}} & \sqrt{\dfrac{l-m+\frac12}{2l+1}} \\ -\sqrt{\dfrac{l-m+\frac12}{2l+1}} & \sqrt{\dfrac{l+m+\frac12}{2l+1}} \end{pmatrix}.$$

!

• Multiplying $\langle\hat n|$.

$\rightarrow$ "spin-angular functions"

$\langle\hat n|l,m-\tfrac12\rangle\otimes|\tfrac12,\tfrac12\rangle$

$$\mathcal{Y}_l^{j=l\pm\frac12,m} = \pm\sqrt{\frac{l\pm m+\frac12}{2l+1}}\, Y_l^{m-\frac12}(\theta,\phi)\,\chi_\uparrow + \sqrt{\frac{l\mp m+\frac12}{2l+1}}\, Y_l^{m+\frac12}(\theta,\phi)\,\chi_\downarrow$$

• ☆ Simultaneous eigenfunctions of $\vec L^2, \vec S^2, \vec J^2, J_z$.

$\rightarrow$ eigenfunction of $\vec L\cdot\vec S = \frac12(\vec J^2 - \vec L^2 - \vec S^2)$

eigenvalue =

$$\frac{\hbar}{2}\left[j(j+1) - l(l+1) - \frac34\right] = \begin{cases} \dfrac{l\hbar^2}{2} & \text{for } j = l+\frac12 \\ -\dfrac{(l+1)\hbar^2}{2} & \text{for } j = l-\frac12 \end{cases},$$

Contributing to the fine structure.

⑤ CG coefficients and Rotation Matrices for $\vec{J} = \vec{J}_1 + \vec{J}_2$

$$\mathcal{D}^{(j_1)}(R) \otimes \mathcal{D}^{(j_2)}(R) = \begin{pmatrix} \boxed{\mathcal{D}^{(|j_1-j_2|)}} & & & \\ & \boxed{\mathcal{D}^{(|j_1-j_2|+1)}} & & \\ & & \ddots & \\ & & & \boxed{\mathcal{D}^{(j_1+j_2)}} \end{pmatrix}$$

In terms of H-space $\mathcal{E}$

$$\mathcal{E}^{(j_1)} \otimes \mathcal{E}^{(j_2)} = \mathcal{E}^{(|j_1-j_2|)} \oplus \mathcal{E}^{(|j_1-j_2|+1)} \oplus \cdots \oplus \mathcal{E}^{(j_1+j_2)}$$

" Block-diagonal matrix of $\mathcal{D}(R)$ "

↓

• **Clebsch-Gordan Series** :

a. $$\mathcal{D}^{(j_1)}_{m_1 m_1'}(R)\, \mathcal{D}^{(j_2)}_{m_2 m_2'}(R) = \sum_{j,m,m'} C^{j_1 j_2}_{m_1 m_2 ; j m}\, C^{j_1 j_2}_{m_1' m_2' ; j m'}\, \mathcal{D}^{(j)}_{m m'}(R)$$

b. $$\mathcal{D}^{(j)}_{m m'}(R) = \sum_{m_1 m_2} \sum_{m_1' m_2'} C^{j_1 j_2}_{m_1 m_2 ; j m}\, C^{j_1 j_2}_{m_1' m_2' ; j m'}\, \mathcal{D}^{(j_1)}_{m_1 m_1'}(R)\, \mathcal{D}^{(j_2)}_{m_2 m_2'}(R)$$

proof of a.

$|j_1 j_2 ; m_1 m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle$

$\mathcal{D}(R) = \mathcal{D}^{(j_1)} \otimes \mathcal{D}^{(j_2)}$

$$\langle j_1 j_2 ; m_1 m_2 | \mathcal{D}(R) | j_1 j_2 ; m_1' m_2' \rangle$$

$$= \langle j_1 m_1 | \mathcal{D}^{(j_1)}(R) | j_1 m_1' \rangle \langle j_2 m_2 | \mathcal{D}^{(j_2)}(R) | j_2 m_2' \rangle$$

also,

$$= \sum_{jm} \sum_{j'm'} \langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j m \rangle \langle j_1 j_2 ; j m | \mathcal{D}(R) | j_1 j_2 ; j' m' \rangle \langle j_1 j_2 ; j' m' | j_1 j_2 ; m_1' m_2' \rangle$$

$$= \sum_{jm} \sum_{j'm'} C^{j_1 j_2}_{m_1 m_2 ; j m}\, \mathcal{D}^{(j)}_{m m'}(R)\, \delta_{jj'}\, C^{j_1 j_2}_{m_1' m_2' ; j' m'}$$

Thus

$$\mathcal{D}^{(j_1)}_{m_1 m_1'}\, \mathcal{D}^{(j_2)}_{m_2 m_2'} = \sum_{j m m'} C^{j_1 j_2}_{m_1 m_2 ; j m}\, C^{j_1 j_2}_{m_1' m_2' ; j m'}\, \mathcal{D}^{(j)}_{m m'}$$

proof of b is very similar.

$(|j_1 - j_2| \le j \le j_1 + j_2)$

$m = m_1 + m_2$

$m' = m_1' + m_2'$

→ Let's rewrite the CG series in terms of $Y_\ell^m(\theta,\phi)$.

– Spherical Harmonics as Rotation Matrices

pp 205–206 S & N

Consider $|\hat{n}\rangle = \mathcal{D}(R)|\hat{z}\rangle \rightarrow |\hat{n}\rangle = \sum_{\ell m} \mathcal{D}(R)\,|\ell,m\rangle\langle \ell,m|\hat{z}\rangle$

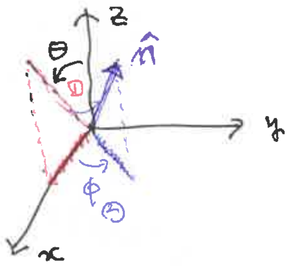
→ multiplying $\langle \ell, m'|$.

$$\langle \ell, m'|\hat{n}\rangle = \sum_m \mathcal{D}^{(\ell)}_{m'm}(R)\,\langle \ell, m|\hat{z}\rangle . \quad \cdots (*)$$

To get (θ, ϕ) in the spherical coordinates,

R can be constructed for $\beta = \theta$, $\alpha = \phi$ $(\gamma = 0)$

$$\mathcal{D}(R) = \mathcal{D}(\alpha=\phi, \beta=\theta, \gamma=0)$$



$$(*) \Rightarrow Y_\ell^{m'*}(\theta,\phi) = \sum_m \mathcal{D}^{(\ell)}_{m'm}(R)\, Y_\ell^{m*}(\theta=0, \phi=\text{undetermined})\, \delta_{m,0}$$

Since $\langle \ell, m|\hat{z}\rangle = Y_\ell^{m*}(\theta=0,\phi)$

$$\propto \frac{1}{\sin^m\theta}\frac{d^{\ell-m}}{d\cos\theta^{\ell-m}}\sin^{2\ell}\theta$$

$$\Rightarrow \sin^m\theta\,[\,\bigcirc\cdot\underbrace{\cos^{\ell-m}\theta}_{\text{highest order in }\cos\theta} + \cdots + \,]$$

$$\xrightarrow{\text{as } \theta\to 0} 0 \quad \text{unless } m=0$$

using $Y_\ell^0(\theta,\phi) = \sqrt{\dfrac{2\ell+1}{4\pi}}\,P_\ell(\cos\theta)$ ⟵⟶ Legendre polynomial.

$$\Rightarrow Y_\ell^{m'*}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\,\mathcal{D}^{(\ell)}_{m',0}(\alpha=\phi, \beta=\theta, \gamma=0).$$

CG series: $D^{(j_1)}_{m_1 m_1'}(R)\, D^{(j_2)}_{m_2 m_2'}(R) = \sum_{j m m'} C^{j_1 j_2}_{m_1 m_2; jm}\, C^{j_1 j_2}_{m_1' m_2'; j m'} \cdot D^{(j)}_{m m'}(R)$

Set $j_1 = l_1$, $j_2 = l_2$, $j = l$

$m_1' = 0$, $m_2' = 0$, so $m' = 0$

$$\Rightarrow \quad Y^{m_1}_{l_1}(\theta,\phi)\, Y^{m_2}_{l_2}(\theta,\phi) = \frac{\sqrt{(2l_1+1)(2l_2+1)}}{4\pi} \cdot$$

$$\sum_{l m} C^{l_1 l_2}_{m_1 m_2; l m}\, C^{l_1 l_2}_{00; l0} \sqrt{\frac{4\pi}{2l+1}}\, Y^m_l(\theta,\phi)$$

multiplying $\int d\Omega\, Y^{m*}_l(\theta,\phi)$.

$$\Rightarrow \int d\Omega\, Y^{m*}_l(\theta,\phi)\, Y^{m_1}_{l_1}(\theta,\phi)\, Y^{m_2}_{l_2}(\theta,\phi)$$

$$= \underbrace{\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\, C^{l_1 l_2}_{00; l0}}_{}\, C^{l_1 l_2}_{m_1 m_2; l m}$$

In dep. of "orientations" (m_1, m_2)

just C-G of $l_1 + l_2 \to l$

: a special case of "Wigner-Eckart" theorem.